Linear Algebra

Inner Product Space

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu
Maryam Ramezani maryam.ramezani@sharif.edu

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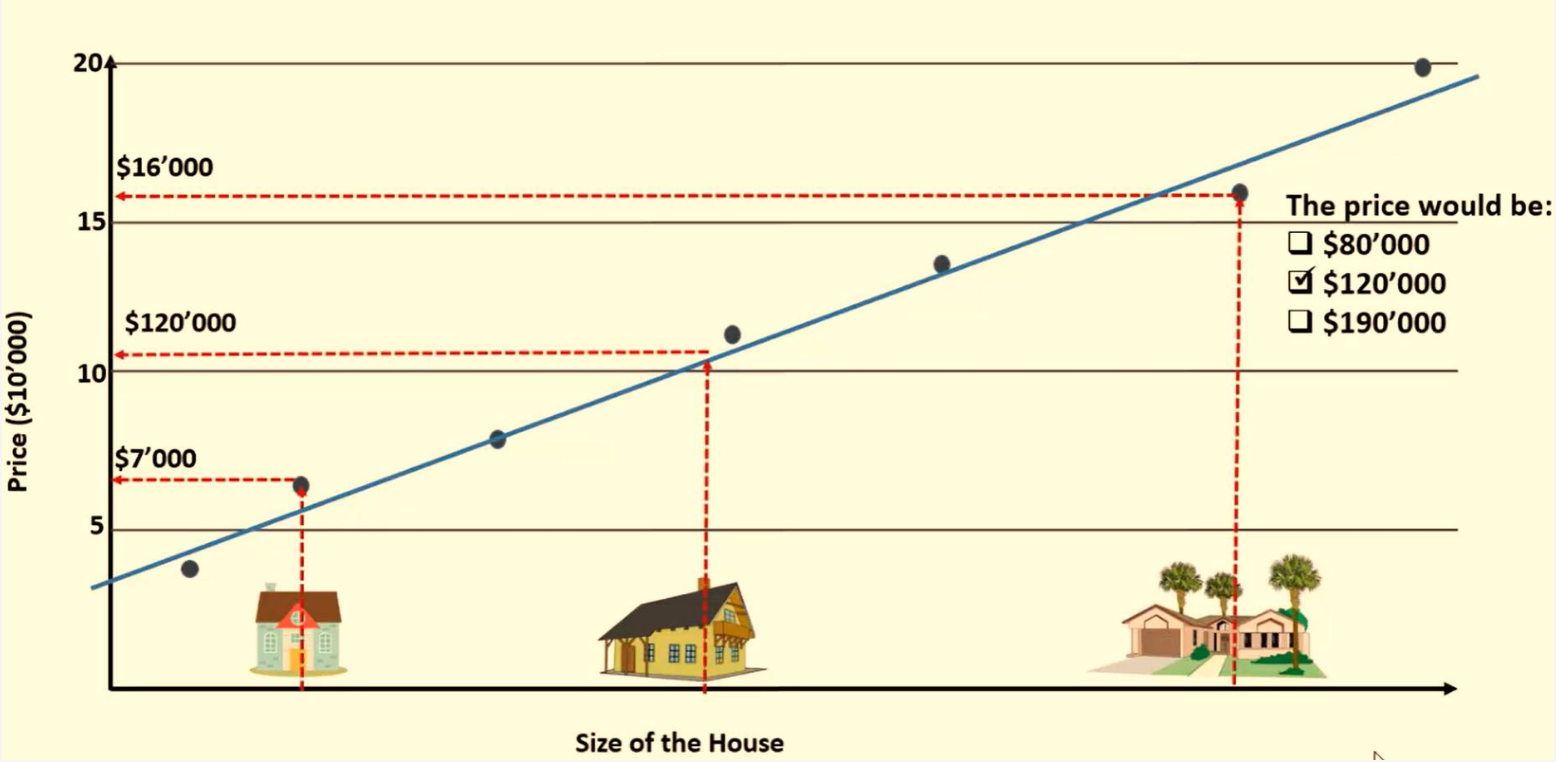
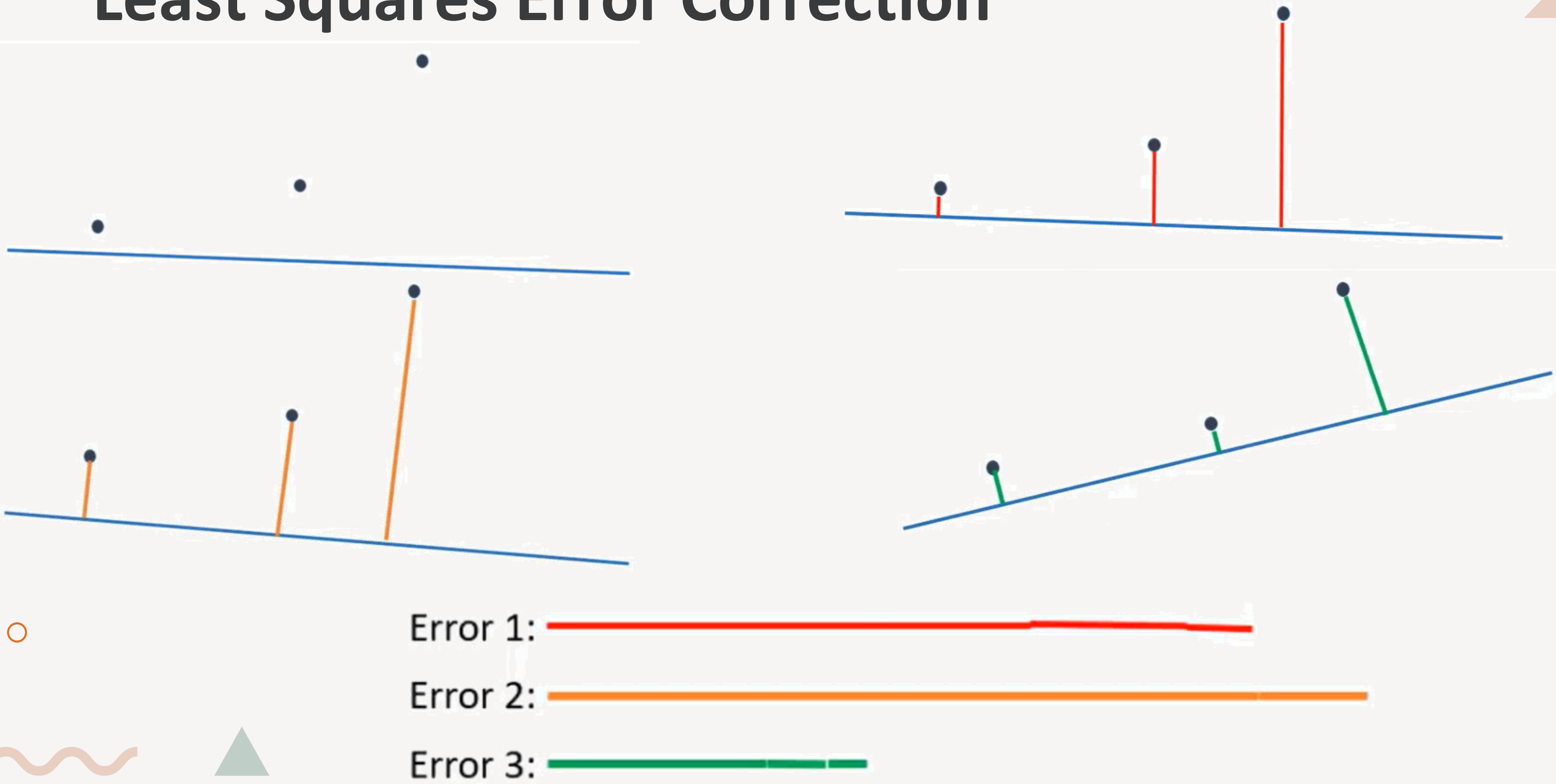
Least Squares Error Correction



Least Squares Error Correction



02

Linear Form

What are Linear Functions?

- $f: R^n \to R$ means that f is a function that maps real n-vectors to real numbers
- $f(x)$ is the value of function f at x (x is referred to as the argument of the function).
- $f(x) = f(x_1, x_2, \ldots, x_n)$: where $x_1, x_2, \ldots, x_n$ are arguments

Definition

A function $f: R^n \to R$ is linear if it satisfies the following two properties:

- **Additivity:** For any n-vector x and y, $f(x + y) = f(x) + f(y)$
- **Homogeneity:** For any n-vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$

Superposition property:

Definition

Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Note

- A function that satisfies the superposition property is called linear

What are Linear Functions?

- If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

Inner product is Linear Function?

Theorem 1

A function defined as the inner product of its argument with some fixed vector is linear.

Proof: $f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

$$1)\ f(x+y) = a^\top(x+y) = a_1(x_1+y_1) + \cdots + a_n(x_n+y_n)$$

$$= a_1x_1 + \cdots + a_nx_n + a_1y_1 + \cdots + a_ny_n = a^\top x + a^\top y = f(x) + f(y)$$

$$2)\ \alpha f(x) = \alpha(a^\top x) = \alpha(a_1x_1 + \cdots + a_nx_n) = a_1\alpha x_1 + \cdots + a_n\alpha x_n$$

$$= a^\top (\alpha x) = f(\alpha x)$$

What are Linear Functions?

Theorem 2

If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.

Proof:

$$f(x) = f(x_1 e_1 + \cdots + x_n e_n) = x_1 f(e_1) + \cdots + x_n f(e_n)$$

$$= \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}^{\top} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}^{\top} x$$

What are Linear Functions?

Theorem 3

The representation of a linear function f in a specific basic as:
$f(x) = a^T x$ is unique, which means that there is only one vector a for which $f(x) = a^T x$ holds for all x.

Proof:

$$f(x) = a^\top x\,, \qquad f(x) = b^\top x$$

$$f(e_1) = a^\top e_1 = a_1\,, \qquad f(x) = b^\top e_1 = b_1 \Rightarrow a_1 = b_1$$
$$f(e_2) = a^\top e_2 = a_2\,, \qquad f(x) = b^\top e_2 = b_2 \Rightarrow a_2 = b_2$$
$$\cdots$$
$$f(e_n) = a^\top e_n = a_n\,, \qquad f(x) = b^\top e_n = b_n \Rightarrow a_n = b_n$$

$$\Rightarrow a = b$$

Linear Form Examples

Example

- Is average a linear function?
- Is maximum a linear function?

$$avg(x) = \frac{x_1 + \cdots + x_n}{n} = \frac{1}{n}x_1 + \cdots + \frac{1}{n}x_n = \begin{pmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix}^\top x$$

$$x = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \max(x) = 5, \qquad y = \begin{pmatrix} 0 \\ 7 \\ 8 \end{pmatrix} \max(y) = 8$$

$$\max(x+y) = 11 \neq 5 + 8 = 13$$

03

Bilinear Form

Bilinear Form over a real vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{F}$. Then a function $f: V \times W \rightarrow \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:

   i. $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and

   ii. $f(c\mathbf{v_1}, \mathbf{w}) = cf(\mathbf{v_1}, \mathbf{w})$ for all $c \in \mathbb{F}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is linear in its second argument:

   i. $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and

   ii. $f(\mathbf{v}, c\mathbf{w_1}) = cf(\mathbf{v}, \mathbf{w_1})$ for all $c \in \mathbb{F}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

Bilinear Form

Note

Let V be a vector space over a field $\mathbb{F}$. Then the **dual** of V, denoted by V^*, is the vector space consisting of all linear forms on V.

Example

Let V be a vector space over a field $\mathbb{F}$. Show that the function $g: V^* \times V \rightarrow \mathbb{F}$ defined by

$$g(f, \mathrm{v}) = f(\mathrm{v}) \text{ for all } f \in V^*, \mathrm{v} \in V$$

is a bilinear form.

Example Solution!

- $g(f_1 + f_2, v) = (f_1 + f_2)v = f_1(v) + f_2(v) = g(f_1, v) + g(f_2, v)$
- $g(cf, v) = (cf)(v) = cf(v) = cg(f, v)$
- $g(f, v_1 + v_2) = f(v_1) + f(v_2) = g(f, v_1) + g(f, v_2)$
- $g(f, cv) = f(cv) = cf(v) = cg(f, v)$

Symmetric Bilinear Form

Definition

A **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V is called symmetric if for all $v, w \in V$:

$$f(v, w) = f(w, v)$$

Bilinear Form arises from a matrix

Theorem 4

Every **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V arises from a matrix for all $v, w \in V$:

$$f(v,w) = v^T A w$$

Proof?

$$f(v,w) = f\left(\sum_i v_i e_i, \sum_j w_j e_j\right) = \sum_i \sum_j v_i w_j f(e_i, e_j)$$

$$= \sum_i \sum_j v_i a_{ij} w_j = v^\top A w$$

Associated Matrices

Definition

If V is a finite-dimensional vector space, $B = \{b_1, \dots, b_n\}$ is a basis of V, and $f: V \times V \to \mathbb{F}$ be a **bilinear form** function the associated matrix A of f with respect to B is the matrix $[f]_B \in \mathbb{F}^{n\times n}$ whose (i, j)-entry is the value $f(b_i, b_j)$.

$$f(v, w) = v^T A w = v^T [f]_B w$$

$$[f]_{\mathcal{B}} = \begin{pmatrix} f(b_1, b_1) & \dots & f(b_1, b_n) \\ \vdots & & \vdots \\ f(b_n, b_1) & \dots & f(b_n, b_n) \end{pmatrix}$$

Associated Matrices

Note

The associated matrix changes if we use a different basis.

Example

For the bilinear form $f\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = 2ac + 4ad - bc$ on $\mathbb{F}^2$, find $[f]_B$ for basis $B = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}\right\}$ and $[f]_P$ for basis P= $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$[f]_B = \begin{bmatrix} 14 & 29 \\ -16 & 20 \end{bmatrix}$ $[f]_P = \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}$

04

Bilinear Form Over Complex Vector Space

Bilinear Form over a complex vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{C}$. Then a function $f: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:

i. $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and

ii. $f(\lambda \mathbf{v_1}, \mathbf{w}) = \lambda f(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is conjugate linear in its second argument:

i. $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and

ii. $f(\mathbf{v}, \lambda \mathbf{w_1}) = \bar{\lambda} f(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

Bilinear Form over a complex vector space

Bilinear forms on $\mathbb{R}^n$	Bilinear forms on $\mathbb{C}^n$
Linear in the first variable	Linear in the first variable
Linear in the second variable	Conjugate linear in the second variable

05

Inner Product

Inner product over real vector space

Definition

An inner product is a positive-definite symmetric bilinear form.

An inner product on V is a function $\langle , \rangle : V \times V \rightarrow \mathbb{R}$ such that $v, w \in V, c \in \mathbb{R}$:

1. $\langle v, v \rangle = 0$ if and only if $v = 0$.
2. $\langle w, v \rangle = \langle v, w \rangle$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, v \rangle \geq 0$ for all $v \in V$.

Inner Product

Why for bilinear form I wrote just two properties instead of four properties?

- Using properties (2) and (4) and again (2)

$$\langle w, cu \rangle = \langle cu, w \rangle = c\langle u, w \rangle = c\langle w, u \rangle$$

- Using properties (2), (3) and again (2)

$$\langle w, u+v \rangle = \langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = \langle w, u \rangle + \langle w, v \rangle$$

1. $\langle v, v \rangle = 0$ if and only if $v = 0$.
2. $\langle w, v \rangle = \langle v, w \rangle$.
3. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, v \rangle \geq 0$ for all $v \in V$.

Inner Products

Note

- For $v \in V, \langle 0, v \rangle = 0$, $\langle v, 0 \rangle$ =0.

$$\langle 0, v \rangle = \langle 0u, v \rangle = 0\langle u, v \rangle = 0$$

General Inner product

Definition

Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that V is a vector space over $\mathbb{F}$. Then an **inner product** on V is a function $\langle \cdot,\cdot \rangle : V \times V \rightarrow \mathbb{F}$ such that the following three properties hold for all $c \in \mathbb{F}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$:

a) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (conjugate symmetry)

b) $\langle \mathbf{v}+c\mathbf{x},\mathbf{w} \rangle=\langle \mathbf{v},\mathbf{w} \rangle+c\langle \mathbf{x},\mathbf{w} \rangle$ (linearity)

c) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$. (pos. definiteness)

Inner Products for vectors

Note

- The standard inner product between vectors is: $(x, y \in \mathbb{R}^n)$

$$\langle x, y\rangle = x^T y = \sum x_i y_i$$

- The function $\langle\cdot,\cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle \mathrm{v}, \mathrm{w}\rangle = v^* \mathrm{w} = \sum_{i=1}^{n} \overline{v_i}\, w_i$$

for all $\mathrm{v}, \mathrm{w} \in \mathbb{C}^n$ is an inner product on $\mathbb{C}^n$.

Inner Products for matrices

Note

The standard inner product between two matrices is: $(X, Y \in \mathbb{R}^{m\times n})$

$$\langle X, Y\rangle = trace(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

Example

Find the inner products of following matrices:

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\langle U, V\rangle = trace(U^\top V) = trace\left(\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}\right) = 0$$

Inner Product for functions

Note

Let $a < b$ be real numbers and let $C[a, b]$ be the vector space of continuous functions on the real interval $[a, b]$. The function $\langle \cdot,\cdot \rangle : C[a, b] \times C[a, b] \to \mathbb{R}$ defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx \qquad \text{for all} \qquad f, g \in C[a, b]$$

is and inner product on $C[a, b]$.

Inner Product for polynomials

Note

- For $p(x)$ and $q(x)$ with at most degree n:

$$\langle p(x), q(x)\rangle = \mathrm{p(0)q(0) + p(1)q(1) + \cdots + p(n)q(n)}$$

- For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \mathrm{p(0)q(0)} + \int_{-1}^{1} p'q'$

- For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}dx$

06

Inner Product Space

Inner product space

Definition

An inner product space is a finite-dimensional real or complex vector space V along with an inner product on V.

real → Euclidean Space

complex → Unitary Space

Resources

- Chapter 8: Kenneth Hoffman and Ray A. Kunze. Linear Algebra. PHI Learning, 2004.
- Chapter 6: Sheldon Axler, Linear Algebra Done Right, 2024.
- Chapter 1: David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Pearson, 2016.
- Chapter 2: David Poole, Linear Algebra: A Modern Introduction. Cengage Learning, 2014.
- Chaper1: Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016.